

# Global solutions in gravity.

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## Abstract

The method of conformal blocks for construction of global solutions in gravity for a two-dimensional metric having one Killing vector field is described.

A space-time in gravity models is a differentiable manifold which has to satisfy at least two requirements. Firstly, the Lorentz signature metric is to be given on it. Secondly, the manifold has to be maximally extended along extremals. The last requirement means that any extremal can be either continued to infinite value of the canonical parameter in both directions or at a finite value of the canonical parameter it ends up at a singular point where one of the geometric invariants, for example, the scalar curvature becomes infinite. Space-time with a given metric satisfying both requirements is called a global solution in gravity. The Kruskal–Szekeres extension [1, 2] of the Schwarzschild solution is a well known example of nontrivial global solution in general relativity.

In general relativity only a small number of global solutions is known in account of the complicated equations of motion (see review [3]). In two-dimensional gravity models attracting much interest last years the situation is simpler, and all global solutions were found [4] in two-dimensional gravity with torsion [5], and in a large class of dilaton gravity models too [6–8]. Constructive method of conformal blocks was proposed [4] for two-dimensional gravity with torsion in the conformal gauge. The use of Eddington–Finkelstein coordinates allowed to prove the smoothness of global solutions [6]. In general relativity solutions having the form of a warped product of two surfaces can also be explicitly constructed and classified [9]. This allowed one to give physical interpretation to many solutions known before only locally. That is, vacuum solutions to the Einstein equations with cosmological constant describe black holes, wormholes, cosmic strings, domain walls, and other solutions of physical interest.

Consider a plane with Cartesian coordinates  $\tau, \sigma$ . Let conformally flat metric of Lorentz signature be given

$$ds^2 = |N(q)|(d\tau^2 - d\sigma^2), \quad N \in C^l, \quad l \geq 2. \quad (1)$$

Let the argument  $q$  to depend on one coordinate only through an ordinary differential equation

$$|dq/d\zeta| = \pm N(q), \quad (2)$$

with the following sign rule

$$\begin{aligned} N > 0 : \quad \zeta = \sigma, \quad \text{sign} + (\text{static}), \\ N < 0 : \quad \zeta = \tau, \quad \text{sign} - (\text{homogeneous}). \end{aligned} \quad (3)$$

Conformal factor may have zeroes and singularities in a finite number of points  $q_i$ ,  $i = 1, \dots, k$  with  $q_1 = -\infty$  and  $q_k = \infty$ . We consider power behavior of the conformal factor near  $q_i$

$$|q_i| < \infty : \quad N(q) \sim (q - q_i)^m, \quad (4)$$

$$|q_i| = \infty : \quad N(q) \sim q^m. \quad (5)$$

At  $|q_i| < \infty$  for positive  $m > 0$  the conformal factor equals zero and defines horizons of a space-time. Negative values  $m < 0$  correspond to singularities. At infinite points positive and negative

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values of  $m$  correspond conversely to singularities and zeroes of the conformal factor. Formulae (1) and (2) define four different metrics, due to the modulus sign in Eq.(2):

$$\begin{aligned} \text{I :} & \quad N > 0, \quad dq/d\sigma > 0, \\ \text{II :} & \quad N < 0, \quad dq/d\tau < 0, \\ \text{III :} & \quad N > 0, \quad dq/d\sigma < 0, \\ \text{IV :} & \quad N < 0, \quad dq/d\tau > 0. \end{aligned} \tag{6}$$

The Schwarzschild metric is an example of the type (1). Indeed, in static domains I and III make the coordinate transformation  $\tau, \sigma \rightarrow \tau, q$

$$ds^2 = N(q)d\tau^2 - \frac{dq^2}{N(q)}, \quad N > 0. \tag{7}$$

Neglecting the angular part in the Schwarzschild metric one gets the metric (7) for

$$N(q) = 1 - \frac{2M}{q}, \tag{8}$$

where  $M$  is the mass of the black hole, and  $q$  is the radius.

The scalar curvature is the same in all four domains  $R = -N''$ . It is singular near  $q_i$  for the following exponents in asymptotic behavior (4):

$$\begin{aligned} |q_i| < \infty : & \quad m < 0, \quad 0 < m < 1, \quad 1 < m < 2, \\ |q_i| = \infty : & \quad m > 2. \end{aligned} \tag{9}$$

At  $q \rightarrow \pm\infty$  the scalar curvature tends to a nonzero constant for  $m = 2$  and to zero for  $m < 2$ .

To construct a global solution we introduce the notion of the conformal block corresponding to every interval  $(q_i, q_{i+1})$ . For definiteness we consider static solution of type I. Then the time coordinate takes values on the whole real axis  $\tau \in R$ . The domain of  $\sigma$  is defined by Eq. (2) which may be rewritten as

$$\sigma \sim \int^{q_i, q_{i+1}} \frac{dq}{N(q)} \tag{11}$$

This integral converge or diverge depending on the exponent  $m$ :

$$\begin{aligned} |q_i| < \infty : & \quad \begin{cases} m < 1, & \text{converge, line,} \\ m \geq 1, & \text{diverge, angle,} \end{cases} \\ |q_i| = \infty : & \quad \begin{cases} m \leq 1, & \text{diverge, angle,} \\ m > 1, & \text{converge, line.} \end{cases} \end{aligned} \tag{12}$$

At the right of this table the form of the boundary of the corresponding conformal blocks is given. If at both ends of the interval  $(q_i, q_{i+1})$  the integral diverge, then  $\sigma \in (-\infty, \infty)$ , and the metric is defined on the whole  $\tau, \sigma$  plain. If at one of the ends  $q_{i+1}$  or  $q_i$  the integral converge, then the metric is defined on the half plain  $\sigma \in (-\infty, \sigma_{i+1})$  or  $\sigma \in (\sigma_i, \infty)$ , correspondingly. If the integral converge on both sides of the interval, then the solution is defined on the strip  $\sigma \in (\sigma_i, \sigma_{i+1})$ . Next we map the  $\tau, \sigma$  plain on the square along the light like directions  $\xi = \tau + \sigma$ ,  $\eta = \tau - \sigma$  using a conformal transformation  $u = u(\xi)$ ,  $v = v(\eta)$  with bounded functions. Then the static solution defined on the whole  $\tau, \sigma$  plain corresponds to a square conformal block. If solution of equation (2) is defined on a half interval, then a static solution corresponds to a triangular conformal block. When solutions of equation (2) is defined on a finite interval the conformal block is represented in the form of a lens. There are two conformal blocks for every interval because equation (2) is invariant under the space reflection  $\sigma \leftrightarrow -\sigma$ .

A detailed analysis of extremals [11] allows one to describe all possible boundaries of conformal blocks summarized in Fig. 1. For definiteness we show the right boundary of static conformal blocks I. Time like boundary is shown by a vertical line and light like boundary by an angle. If the scalar curvature is singular on the boundary (9), (10), then it has protuberances. Incomplete and complete boundaries are shown by dashed and thick solid lines, respectively. Exclusion is the semicomplete boundary corresponding to  $|q_i| = \infty$  for  $m > 2$  (see Fig. 1). Light like extremals reaching this boundary are complete while space like ones are not. Lower and upper corners of all static conformal blocks (time past and future infinities) are essentially singular points and always complete. Completeness and incompleteness of right corners of angle boundaries are shown

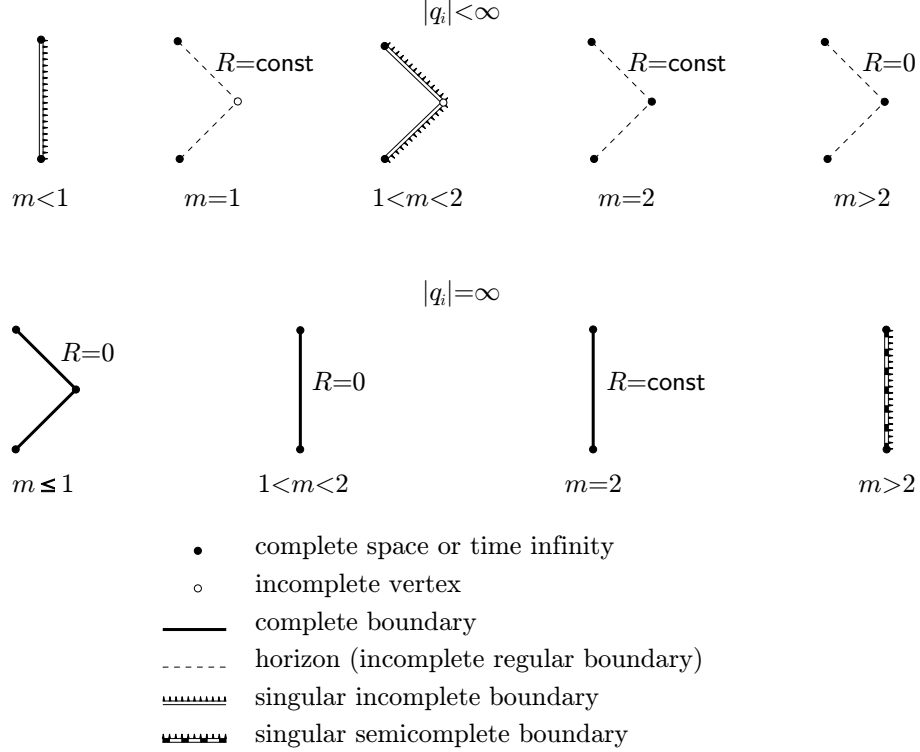


Figure 1: The form of the right boundary of a static conformal block.

by filled and not filled circles, respectively. As a result only horizons correspond to incomplete boundaries, the scalar curvature being finite on them. Therefore solutions of the form (1) must be continued only through horizons. Let us formulate the rules and the theorem for construction of global solutions.

- 1) Every global solution for a metric (1) corresponds to the interval of the variable  $q \in (q_-, q_+)$ , where  $q_{\pm}$  are either infinite points or a curvature singularities defined by the condition (9). Singularities inside the interval must be absent.
- 2) If there are no zeroes inside the interval  $(q_-, q_+)$  then the corresponding conformal block is the maximally extended solution.
- 3) If there are zeroes (horizons) inside the interval  $(q_-, q_+)$  then enumerate them,  $N(q_j) = 0$ ,  $j = 1, \dots, n$ ,  $q_j \in (q_-, q_+)$ , and associate with each of the intervals  $(q_-, q_1), \dots, (q_n, q_+)$  a pair of static or homogeneous conformal blocks for  $N > 0$  and  $N < 0$ , respectively.
- 4) Sew together conformal blocks along horizons  $q_j$ , preserving the smoothness of the conformal factor, that is sew together conformal blocks corresponding only to adjacent intervals  $(q_{j-1}, q_j)$  and  $(q_j, q_{j+1})$ , and if the gluing is performed for static or homogeneous conformal blocks, then sew together blocks of one type.
- 5) The Carter–Penrose diagram obtained by gluing all adjacent conformal blocks is a connected fundamental region if inside the interval  $(q_-, q_+)$  the conformal factor changes its sign. If  $N \geq 0$  or  $N \leq 0$  everywhere inside the interval  $(q_-, q_+)$  then one gets two fundamental regions related by space or time reflection.
- 6) For one zero of an odd degree the boundary of the fundamental region consists of boundaries of the conformal blocks corresponding to the points  $q_-$  and  $q_+$ , and the Carter–Penrose diagram represents the global solution.
- 7) If there is one zero of even degree or two or more zeroes of arbitrary degree the boundary of the fundamental region includes horizons, and it has to be either continued periodically in space and (or) time or the opposite sides should be identified.
- 8) If the fundamental group of the Carter–Penrose diagram is trivial then it is the universal covering space for a global solution.
- 9) If the fundamental group of the Carter–Penrose diagram is nontrivial, then construct the corresponding universal covering space.

**Theorem.** The universal covering space constructed according to the rules 1–9 is the maximally

extended pseudo riemannian  $C^{l+1}$  manifold with the continuous  $C^l$ , metric such that every point not lying on a horizon has a neighborhood isometric to some domain with the metric (1).

The idea of the proof is to go to Eddington–Finkelstein coordinates on a horizon and to Kruskal–Szekeres coordinates near a saddle point [11]. All other global solutions are obtained as factor spaces of the universal covering space by a discrete transformation group. For example, one may identify similar horizons lying on a boundary of a fundamental region [4, 10].

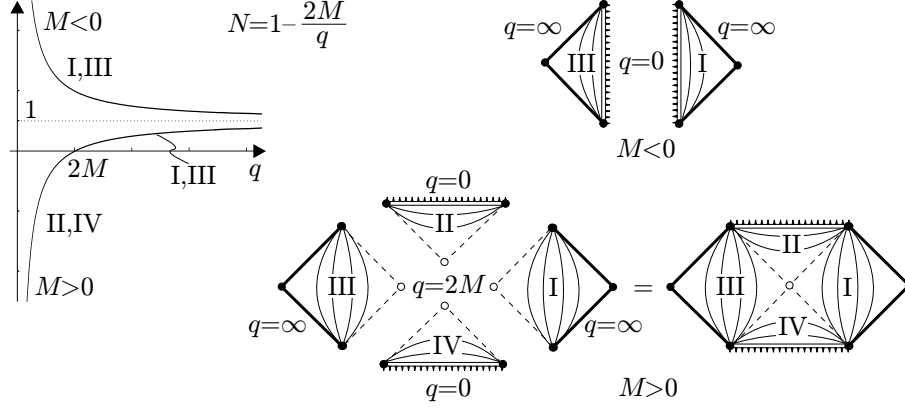


Figure 2: The conformal factor and Carter–Penrose diagrams for the Schwarzschild solution.

To illustrate the conformal blocks method we consider the Schwarzschild solution (8) with  $M > 0$ , Fig. 2. It has a simple pole at  $q = 0$  corresponding to a curvature singularity (9). The point  $q_1 = 2M$  is a simple zero and corresponds to a horizon. The values  $q = \pm\infty$  correspond to asymptotically flat space infinity. We see that two global solutions for positive and negative  $q$  correspond to the infinite interval  $q \in (-\infty, \infty)$ . For positive  $q \in (0, \infty)$  one has  $q_- = 0$ ,  $q_1 = 2M$  and  $q_+ = \infty$ . For each interval  $q \in (0, 2M)$  and  $q \in (2M, \infty)$  there are two homogeneous and static conformal blocks. The boundary elements are defined in Fig. 1. The global solution is uniquely constructed by gluing together four conformal blocks and yields the Kruskal–Szekeres extension of the Schwarzschild solution. For negative  $q$  horizons are absent, and maximally extended solutions are represented by triangular conformal blocks.

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## References

- [1] M.D. Kruskal, Phys. Rev. 119 (1960) 1743.
- [2] G. Szekeres, Publ. Mat. Debrecen. 7 (1960) 285.
- [3] B. Carter, Black hole equilibrium states. In C. DeWitt and B.C. DeWitt (eds.), Black Holes, Gordon & Breach, New York, 1973.
- [4] M.O. Katanaev, J. Math. Phys. 34 (1993) 700.
- [5] M.O. Katanaev and I.V. Volovich, Phys. Lett. 175B (1986) 413.
- [6] T. Klösch and T. Strobl, Class. Quantum Grav. 13 (1996) 2395.
- [7] M.O. Katanaev, W. Kummer, and H. Liebl, Phys. Rev. D53 (1996) 5609.
- [8] M.O. Katanaev, W. Kummer, and H. Liebl, Nucl. Phys. B486 (1997) 353.
- [9] M.O. Katanaev, T. Klösch, and W. Kummer, Ann. Phys. 276 (1999) 191.
- [10] T. Klösch and T. Strobl, Class. Quantum Grav. 14 (1997) 1689.
- [11] M.O. Katanaev, gr-qc/9907088.